REGULAR THERMAL REGIME IN A MULTI-LAYER MEDIUM

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Inzhenerno-fizicheskii zhurnal, Vol. 8, No. 1, pp. 72-78, 1965

Regular thermal regime is examined for a multi-layer medium with perfect and imperfect contact. Possible experimental methods of investigating such a system are determined.

Let us suppose that appreciable variation of thermal conductivity occurs in a certain medium. Then it is rational to treat the medium as if it were composed of many layers with a constant thermal diffusivity $a_i = \lambda_i/\rho_i c_i$. The conduction equations for each layer may be written in the form

$$\lambda_i \frac{\partial^2 u^{(i)}}{\partial x^2} = c_i \rho_i \frac{\partial u^{(i)}}{\partial t}, \tag{1}$$

where the temperature $u^{(i)}(x, t)$ depends on time t and the coordinate x.

Assume that the layers are differentiated by the x values

$$x = x_0,\ x = x_1, \ldots,\ x = x_n\ (x_0 < x_1 < \ldots < x_n).$$

We also have the boundary conditions and conditions defining perfect contact:

$$\begin{gathered} u^{(1)}(x_0, t) = A = \text{const}, \\ u^{(i)}(x_i, t) = u^{(i+1)}(x_i, t), \\ \lambda_i \frac{\partial}{\partial x} u^{(i)}(x_i, t) = \lambda_{i+1} \frac{\partial}{\partial x} u^{(i+1)}(x_i, t), \\ i = 1,2, \ldots, n-1, \\ u^{(n)}(x_n, t) = B = \text{const}, \end{gathered} \tag{2}$$

and the initial conditions

$$u^{(i)}(x, 0) = \psi^{(i)}(x). \tag{3}$$

We shall seek a solution using the Doetsch integral transform [1], where the kernels satisfy the following equation [2]:

$$\lambda_i K_j^{(i)''}(x) + \mu_j^2 c_i \rho_i K_j^{(i)}(x) = 0, \tag{4}$$

with boundary conditions

$$\begin{gathered} K_j^{(1)}(x_0) = 0,\ K_j^{(i)}(x_i) = K_j^{(i+1)}(x_i), \\ \lambda_i K_j^{(i)'}(x_i) = \lambda_{i+1} K_j^{(i+1)'}(x_i),\ i = 1,2, \ldots, n-1,\quad K_j^{(n)}(x_n) = 0. \end{gathered} \tag{5}$$

Solutions of (4) will be

$$K_j^{(i)}(x) = M_j^{(i)} \sin \mu_j \varkappa_i (x_i - x) + N_j^{(i)} \cos \mu_j \varkappa_i (x_i - x),$$

where the constants $M_j^{(i)}$, $N_j^{(i)}$ are determined from (5) and $\varkappa_i = 1/a_i$.

The determinant of this system, equated to zero, gives the characteristic equation, which is a transcendental equation for determining the eigenvalues μ_j^2. One of the coefficients $M_j^{(i)}$, $N_j^{(i)}$ can be so chosen that the functions $K_j(x) = K_j^{(i)}(x)$ for $x_{i-1} < x < x_i$ are normalized.

We must verify that functions $K_j(x)$ will be orthogonal with weight $c(x)\rho(x) = c_i \rho_i$ when $x_{i-1} < x < x_i$ on $[x_0, x_n]$. Applying the integral transform to equations (1), we obtain in the region in question

$$\bar{u}_j'(t) + \mu_j^2 \bar{u}_j(t) = F_j,$$

where

$$\begin{gathered} F_j = \lambda_1 A K_j^{(1)'}(x_0) - \lambda_n B K_j^{(i)'}(x_n), \\ \bar{u}_j(t) = \int_{x_0}^{x_n} c(x)\rho(x) K_j(x) u(x, t) dx, \\ u(x, t) = u^{(i)}(x, t) \text{ for } x_{i-1} < x < x_i. \end{gathered} \tag{6}$$

The initial condition (3) in the mapping region gives

$$\bar{u}_j(0) = \bar{\psi}_j. \tag{7}$$

The solution of (6) in conjunction with conditions (7) gives

$$\bar{u}_j(t) = [\bar{\psi}_j - F_j/\mu_j^2]\exp[-\mu_j^2 t] + F_j/\mu_j^2.$$

The solution of the problem set out in (1)-(3) is given by the series in orthogonal transformation kernels

$$u^{(i)}(x,\ t) = \sum_{j=0}^{\infty} \bar{u}_j(t) K_j^{(i)}(x) =$$

$$= \sum_{j=0}^{\infty} \left[\bar{\psi}_j - \frac{F_j}{\mu_j^2} \right] \exp[-\mu_j^2 t] K_j^{(i)}(x) + \sum_{j=0}^{\infty} \frac{F_j}{\mu_j^2} K_j^{(i)}(x). \tag{8}$$

In this solution we shall confine ourselves to the first term in the sum, which contains the exponential function corresponding to the regular regime.

Let $[\bar{\psi}_j - F_j/\mu_j^2] K_j^{(i)}(x) | \leqslant C$.

Writing the solution u(x, t) in the form

$$u^{(i)}(x,\ t) = \left[\bar{\psi}_0 - \frac{F_0}{\mu_0^2} \right] \exp[-\mu_0^2 t] K_0^{(i)}(x) +$$

$$+ \sum_{j=0}^{\infty} \frac{F_j}{\mu_j^2} K_j^{(i)}(x) + R^{(i)}(x,\ t),$$

we evaluate the remainder of the series R(x, t). We have

$$|R^{(i)}(x,\ t)| \leqslant C \sum_{j=1}^{\infty} \exp[-\mu_j^2 t].$$

The time of onset of the regular regime can be established by the method proposed in [3]. If, for example, $\mu_j^2 \geq j^2$, where j = 1, 2, ..., then $|R^{(i)}(x,\ t)| < \varepsilon$, when

$$t > \pi C^2/4\varepsilon^2.$$

The rate of heating $m = \mu_0^2$ will be a constant for the whole medium, and may be determined experimentally by the method set out in [4].

It is of some interest to develop a regular regime theory for multi-layer media with imperfect contact.

Let us examine the problem for two layers. Then the thermal conduction equations for the two layers will take the form

$$a_i \frac{\partial^2 u^{(i)}}{\partial x^2} = \frac{\partial u^{(i)}}{\partial t} + f^{(i)}, \tag{9}$$

$$u^{(1)} = u^{(1)}(x,\ t),\ f^{(1)} = f^{(1)}(x,\ t) \quad \text{for} \quad x_1 \leqslant x < \zeta;$$

$$u^{(2)} = u^{(2)}(x,\ t),\ f^{(2)} = f^{(2)}(x,\ t) \quad \text{for} \quad \zeta < x \leqslant x_2;\ 0 \leqslant t < \infty.$$

Functions $f^{(i)}$ give the heat source distribution to within a constant multiplier.

At the outer boundaries let there be free heat transfer with a region of variable temperature (boundary conditions of the third kind)

$$\alpha_1 \frac{\partial}{\partial x} u^{(1)}(x_1,\ t) + \beta_1 u^{(1)}(x_1,\ t) = \varphi_1(t), \tag{10}$$

$$\alpha_3 \frac{\partial}{\partial x} u^{(2)}(x_2,\ t) + \beta_3 u^{(2)}(x_2,\ t) = \varphi_3(t), \tag{11}$$

and at the interface let there be incomplete contact

$$\beta_{12} u^{(1)}(\zeta,\ t) + \beta_{22} u^{(2)}(\zeta,\ t) = \varphi_{12}(t), \tag{12}$$

$$\alpha_{12}\frac{\partial}{\partial x}u^{(1)}(\zeta,\ t)+\alpha_{22}\frac{\partial}{\partial x}u^{(2)}(\zeta,\ t)=\varphi_{22}(t). \tag{13}$$

The initial conditions may be taken arbitrarily:

$$u^{(1)}(x,\ 0)=\psi^{(1)}(x), \tag{14}$$

$$u^{(2)}(x,\ 0)=\psi^{(2)}(x). \tag{15}$$

We shall seek a solution in the form of a series in eigenfunctions of the following problem [5]:

$$\frac{1}{\alpha_{22}\beta_{22}}K_j^{(1)''}(x)+\mu_j^2L_1K_j^{(1)}(x)=0, \tag{16}$$

$$\frac{1}{\alpha_{12}\beta_{12}}K_j^{(2)''}(x)+\mu_j^2L_2K_j^{(2)}(x)=0, \tag{17}$$

where

$$L_1=\frac{1}{\alpha_{22}\beta_{22}a_1},\quad L_2=\frac{1}{\alpha_{12}\beta_{12}a_2};$$

with boundary conditions as follows:

$$\begin{aligned}&\alpha_1K_j^{(1)'}(x_1)+\beta_1K_j^{(1)}(x_1)=0,\\&\beta_{12}K_j^{(1)}(\zeta)+\beta_{22}K_j^{(2)}(\zeta)=0,\\&\alpha_{12}K_j^{(1)'}(\zeta)+\alpha_{22}K_j^{(2)'}(\zeta)=0,\\&\alpha_3K_j^{(2)'}(x_2)+\beta_3K_j^{(2)}(x_2)=0.\end{aligned} \tag{18}$$

The solution of (16) and (17) with boundary conditions (18) will be

$$\begin{aligned}K_j^{(1)}(x)&=A_j^{(1)}\sin\mu_j\varkappa_1(\zeta-x)+B_j^{(1)}\cos\mu_j\varkappa_1(\zeta-x),\\K_j^{(2)}(x)&=A_j^{(2)}\sin\mu_j\varkappa_2(x_2-x)+B_j^{(2)}\cos\mu_j\varkappa_2(x_2-x),\end{aligned} \tag{19}$$

where $\varkappa_1^2=\alpha_{22}\beta_{22}L_1$, $\varkappa_2^2=\alpha_{12}\beta_{12}L_2$, and the constants $A_j^{(i)}$, $B_j^{(i)}$ satisfy the system:

$$\begin{aligned}&A_j^{(1)}[\beta_1S_j^{(1)}-\alpha_1\mu_j\varkappa_1C_j^{(1)}]+B_j^{(1)}[\beta_1C_j^{(1)}+\alpha_1\mu_j\varkappa_1S_j^{(1)}]=0,\\&B_j^{(1)}\beta_{12}+A_j^{(2)}\beta_{22}S_j^{(2)}+B_j^{(2)}\beta_{22}C_j^{(2)}=0,\\&A_j^{(1)}\alpha_{12}\mu_j\varkappa_1+A_j^{(2)}\alpha_{22}\mu_j\varkappa_2C_j^{(2)}-B_j^{(2)}\alpha_{22}\mu_j\varkappa_2S_j^{(2)}=0,\\&A_j^{(2)}\alpha_3\mu_j\varkappa_2-B_j^{(2)}\beta_3=0,\end{aligned} \tag{20}$$

where

$$S_j^{(1)}=\sin\mu_j\varkappa_1(\zeta-x_1),\quad S_j^{(2)}=\sin\mu_j\varkappa_2(x_2-\zeta),$$

$$C_j^{(1)}=\cos\mu_j\varkappa_1(\zeta-x_1),\quad C_j^{(2)}=\cos\mu_j\varkappa_2(x_2-\zeta).$$

For (19) to be nontrivial, it is necessary that the determinant of (17) equal zero, $\Delta(\mu_j^2)=0$. Hence the eigenvalues of problem (16)-(18) may be determined.

It is easy to verify that the functions

$$K_j(x)=\begin{cases}K_j^{(1)}(x) & \text{for}\quad x_1\leqslant x<\zeta,\\ K_j^{(2)}(x) & \text{for}\quad \zeta<x\leqslant x_2\end{cases}$$

are orthogonal with weight L_i on $[x_1,\ x_2]$.

One of the coefficients $A_j^{(i)}$, $B_j^{(i)}$ is arbitrary, and it may be chosen so as to normalize the system of functions $K_j(x)$.

Then the function

$$u(x,\ t)=\begin{cases}u^{(1)}(x,\ t) & \text{for}\quad x_1\leqslant x<\zeta,\\ u^{(2)}(x,\ t) & \text{for}\quad \zeta<x\leqslant x_2\end{cases}$$

can be expanded in a series in functions $K_j(x)$:

$$u(x, t) = \sum_{j=0}^{\infty} \bar{u}_j(t) K_j(x), \quad (21)$$

where the summation extends over all j, for which eigenvalues μ_j^2 are different, and

$$\bar{u}_j(t) = L_1 \int_{x_1}^{\zeta} K_j^{(1)}(x) u^{(1)}(x, t)\, dx + L_2 \int_{\zeta}^{x_2} K_j^{(2)}(x) u^{(2)}(x, t)\, dx. \quad (22)$$

Transformation (22) is called a Doetsch transform [1].

The corresponding inversion formula is given by series (21). Rewriting equations (9) in the form:

$$\frac{1}{\alpha_{22}\beta_{22}} \frac{\partial^2 u^{(1)}}{\partial x^2} = L_1 \frac{\partial u^{(1)}}{\partial t} + L_1 f^{(1)},$$

$$\frac{1}{\alpha_{12}\beta_{12}} \frac{\partial^2 u^{(2)}}{\partial x^2} = L_2 \frac{\partial u^{(2)}}{\partial t} + L_2 f^{(2)}$$

and transferring them to the mapping region, we have

$$\bar{u}_j'(t) + \mu_j^2 \bar{u}_j(t) = F_j(t),$$

where

$$F_j(t) = \frac{1}{\alpha_{22}\beta_1\beta_{22}} K_j^{(1)'}(x_1)\, \varphi_1(t) + \frac{1}{\alpha_{12}\beta_{12}\beta_{22}} K_j^{(2)'}(\zeta)\, \varphi_{12}(t) +$$

$$+ \frac{1}{\alpha_{12}\alpha_{22}\beta_{22}} K_j^{(1)}(\zeta)\, \varphi_{22}(t) + \frac{1}{\alpha_{12}\alpha_3\beta_{12}} K_j^{(2)}(x_2)\, \varphi_3(t) - \bar{f}_j(t).$$

In the mapping region initial conditions (14), (15) take the form

$$\bar{u}_j(0) = \bar{\psi}_j .$$

Then, in the mapping region, the solution of our problem takes the form

$$\bar{u}_j(t) = \left[\bar{\psi}_j + \int_0^t F_j(\tau) \exp[\mu_j^2 \tau]\, d\tau\right] \exp[-\mu_j^2 t] \quad . \quad (23)$$

The final solution is given in the form of series (21). We shall examine the question of the regular regime. Let functions $\varphi_1(t)$, $\varphi_{12}(t)$, $\varphi_{22}(t)$, $\varphi_3(t)$, $f^{(i)}(x, t)$ increase no faster than $M \exp(-\eta t)$, where $M > 0$ and $0 < \eta < \mu_j^2$ are certain constants [3].

We shall determine the transform of solution of (23). We have

$$|\bar{u}_j(t)| \leqslant |\bar{\psi}_j| \exp[-\mu_j^2 t] + \int_0^t |F_j(\tau)| \exp[\mu_j^2 \tau]\, d\tau \exp[-\mu_j^2 t],$$

or

$$|\bar{u}_j(t)| \leqslant \frac{M_1}{\mu_j^2 - \eta} \exp[-\eta t],$$

where M_1 is a deliberately chosen constant.

Let the transformation kernels be uniformly bounded

$$|K_j(x)| \leqslant K.$$

We write solution (21) in the form

$$u(x, t) = \sum_{j=0}^{\infty} \bar{u}_j(t) K_j(x) = \bar{u}_0(t) K_0(x) + R(x, t).$$

The time to reach the regular thermal regime will depend on the remainder of the series R(x, t).

But

$$|R(x,t)| \leqslant M_1 K \sum_{j=1}^{\infty}\left[\exp[-\mu_j^2 t] + \frac{1}{\mu_j^2 - \eta}\exp[-\eta t]\right]$$

or

$$|R(x,t)| \leqslant N\sum_{j=1}^{\infty}\exp[-\mu_j^2 t] + N\exp[-\eta t]\sum_{j=1}^{\infty}\frac{1}{\mu_j^2-\eta}, \quad N = M_1 K.$$

Let $\mu_j^2 \geq j^2$ [3, 6]. Then the series $\sum_{j=1}^{\infty}(\mu_j^2-\eta)^{-1}$ converges. Denoting its sum by σ_1, we obtain

$$|R(x,t)| \leqslant N\sum_{j=1}^{\infty}\exp[-j^2 t] + N\sigma_1\exp[-\eta t].$$

But

$$\sum_{j=1}^{\infty}\exp[-j^2 t] \leqslant \int_0^{\infty}\exp[-\theta^2 t]\,d\theta = \frac{1}{2}\sqrt{\frac{\pi}{t}},$$

whence

$$|R(x,t)| < \varepsilon,$$

as soon as

$$t > \max\left\{\frac{\pi N^2}{4\varepsilon^2},\ \frac{1}{\eta}\ln\frac{2N\sigma_1}{\varepsilon}\right\}. \tag{24}$$

if $\mu_j^2 = \pi(j-\alpha)^2$, $0 < \alpha < \frac{1}{2}$ [7], then $\sum_{j=1}^{\infty}\frac{1}{\mu_j^2-\eta} = \sigma_2 < \infty$

and

$$\sum_{j=1}^{\infty}\exp[-\mu_j^2 t] < \sqrt{\frac{1}{\pi t}},$$

therefore

$$|R(x,t)| < \varepsilon,$$

when

$$t > \max\left\{\frac{4N^2}{\pi\varepsilon^2},\ \frac{1}{\eta}\ln\frac{2N\sigma_2}{\varepsilon}\right\}. \tag{25}$$

Thus, the time of onset of the regular regime has been determined.

It is clear from (24) and (25) that it is convenient to make η as large as possible. Therefore, if the eigenvalues are renumbered so that $\mu_0^2 < \mu_1^2 < \mu_2^2 \ldots$, then η may be taken between the first two values $\mu_0^2 < \eta < \mu_1^2$, bearing in mind that $M\exp(-\eta t)$ must majorize the boundary functions and the source functions. If these functions tend to zero comparatively slowly, the time to reach the regular thermal regime also increases: the smaller η, the greater t.

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14 April 1964

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